Generating Gaussian fields using the SPDE approach and Radial Basis Functions

Yiannis Andrianakis, Gemma Stephenson and Peter Challenor

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Motivation - outline

- Gaussian Fields as a solution to a SPDE
- Advantages:
  - non-stationary GFs
  - GFs on manifolds
  - oscillating GFs
  - sparse precision matrices
- Finite element solution works well for small input dimension, but it's hard to extend to higher dimensions
- We are investigating the applicability of Radial Basis Functions, that extend more naturally to higher dimensions

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Generating Gaussian fields with the spde approach

In [Lindgren et al. 2011], it was shown that the SPDE:

$$(\kappa^2 - \Delta)^{\alpha/2} u(\mathbf{x}) = \mathcal{W}(\mathbf{x})$$

has as a solution the Gaussian Field

$$u(\mathbf{x}) \sim \mathcal{N}(0, c(\mathbf{x}, \mathbf{x}'))$$

where $c(\mathbf{x}, \mathbf{x}')$ is the Matérn correlation function.

$\mathbf{x} \in \mathcal{R}^d$: input
$u(\mathbf{x})$: GF, spde solution
$\mathcal{W}(\mathbf{x})$: innovation process
$\Delta = \sum_{i=1}^{d} \partial^2 / \partial x_i^2$

κ: inverse correlation length
α, ν: differentiability parameters
$\mathcal{W}(\mathbf{x})$: innovation process
$K_\nu, \Gamma(\nu)$: Bessel, Gamma functions

Non stationary fields

Making κ a function of $\mathbf{x}$, and introducing a scaling parameter $\tau(\mathbf{x})$, the SPDE:

$$(\kappa^2(\mathbf{x}) - \Delta)^{\alpha/2}\tau(\mathbf{x})u(\mathbf{x}) = \mathcal{W}(\mathbf{x})$$

can generate non-stationary fields.

In particular, allowing $\kappa(\mathbf{x})$, $\tau(\mathbf{x})$ to vary slowly, results in a Gaussian field that is locally Matérn.

The local fields are combined to a global consistent field, automatically by the SPDE.

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Solution of the SPDE

- Discretise the input space $\{\mathbf{x}_i\}, i = 1 \ldots n$
- Define a set of basis functions $\phi_i(\mathbf{x})$, one for each point in the discretisation.
- An approximation to the solution $u(\mathbf{x})$ can be written as $\hat{u}(\mathbf{x}) = \Phi\mathbf{w}$
- A weak solution to the spde is given by[1]

$$\{\langle\phi_i, (\kappa^2 - \Delta)^{\alpha/2}\hat{u}\rangle\} = \{\langle\phi_i, W\rangle\}, i = 1 \ldots n$$

- The correlation function of $u(\mathbf{x})$ (Gaussian field) is a function of inner products between $\phi_i(\mathbf{x})$ and $\nabla\phi_i(\mathbf{x})$.

[1] $\langle f, g\rangle = \int f(x)g(x)\mathrm{d}x$ (inner product)

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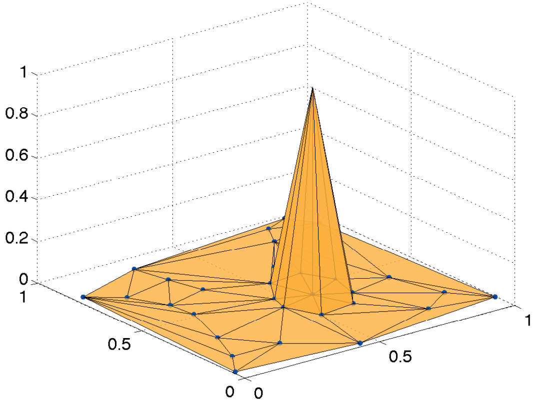
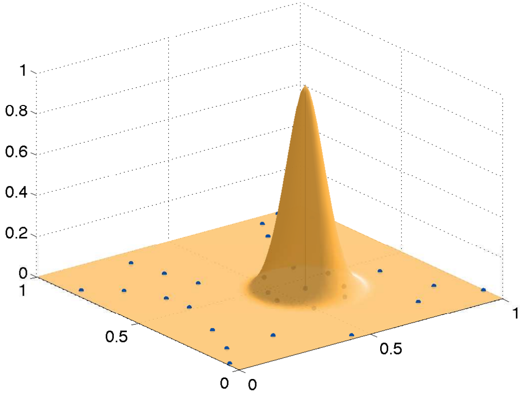
Basis functions (Finite Elements)



Finite elements

- Pros:
  - Vast literature on the topic
  - The precision matrix of $u(\mathbf{x})$ is sparse
- Cons:
  - The triangulation is computationally expensive
  - It is hard to extend to more than 2 input dimensions

Radial basis functions



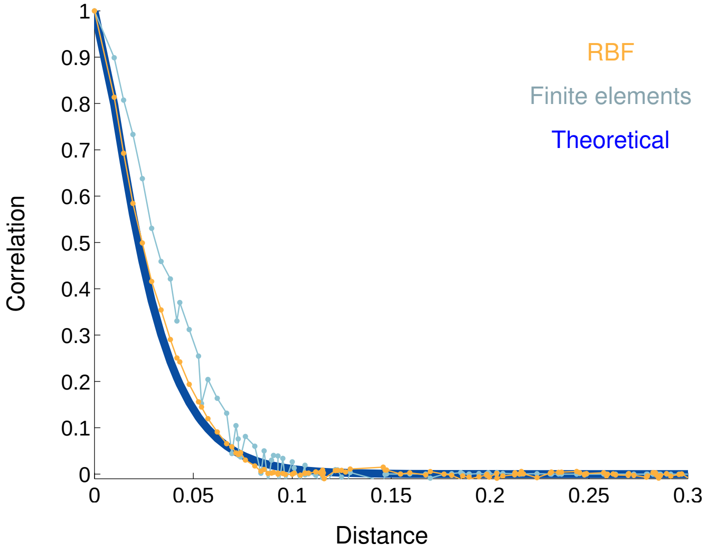
Radial basis functions (2)

- Cons:
  - Area less mature than finite elements
  - Computationally expensive for compactly supported rbfs
  - Presicion matrix is *not* sparse
- Pros:
  - Easier extension to higher dimensions
  - Efficient for not compactly supported rbfs
  - Does not require triangulation.

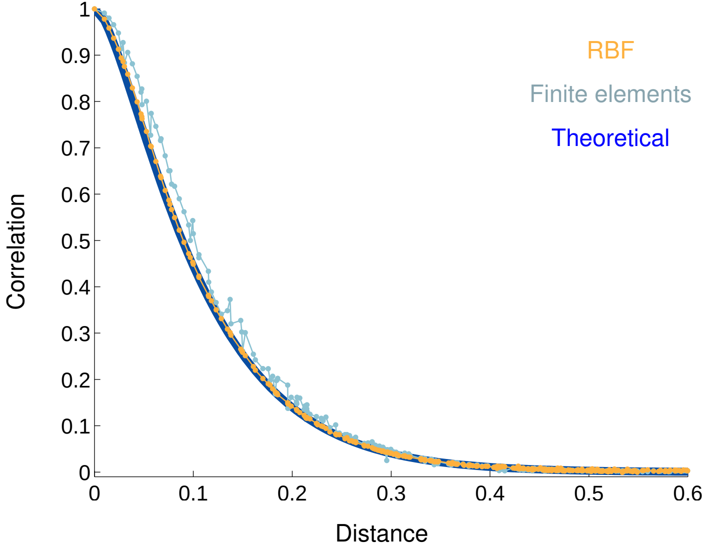
Results

- We generate Gaussian fields in $[0, 1]^d$, for $d = \{2, 6\}$
- We use the stationary SPDE, i.e. known covariance function
- Compare Finite Elements and RBF, with the theoretical solution

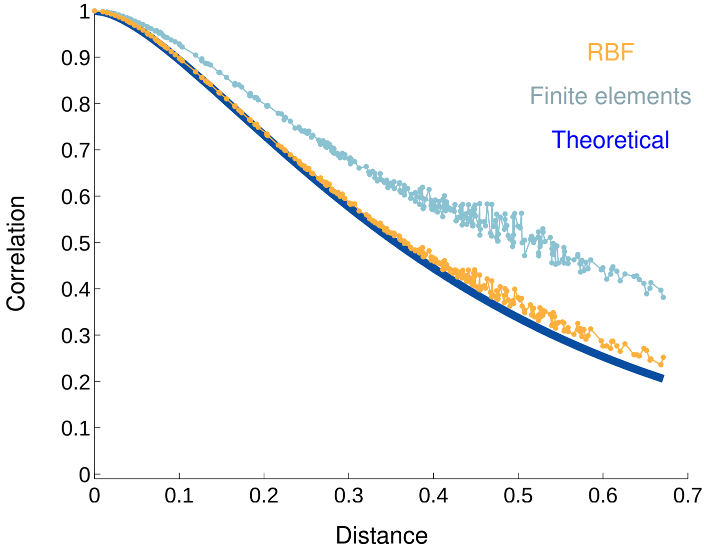
$d = 2$, $n = 300$, $\rho = 0.05$



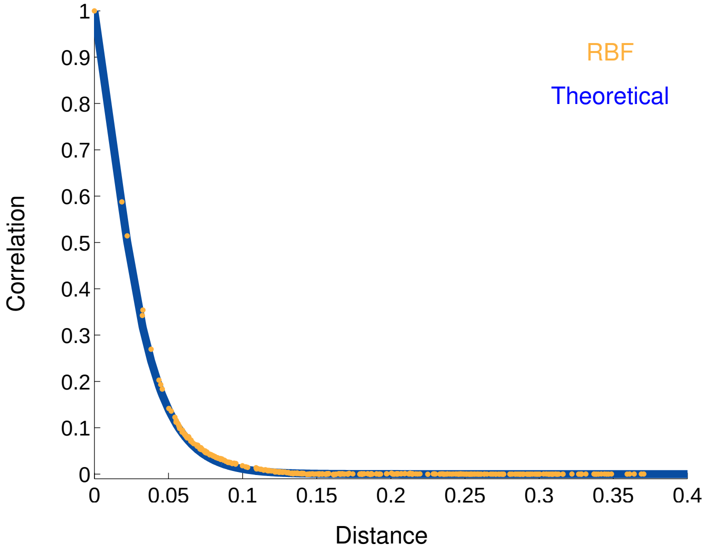
$d = 2$, $n = 300$, $\rho = 0.2$



$d = 2$, $n = 300$, $\rho = 0.8$



$d = 6$, $n = 300$, $\rho = 0.05$



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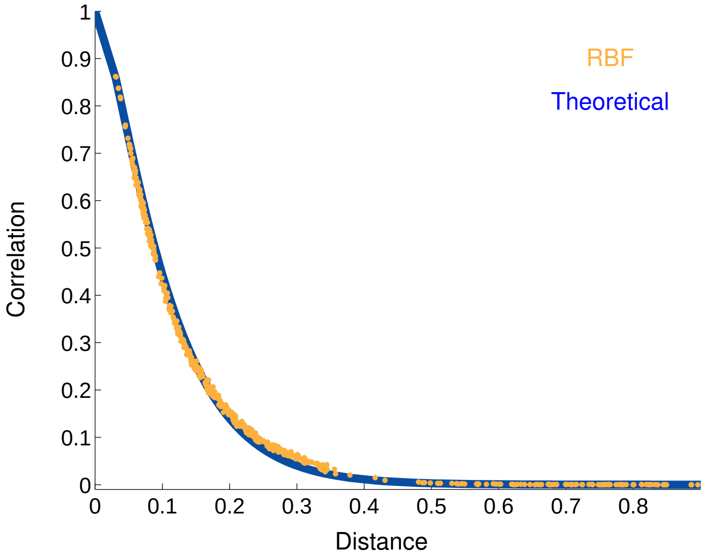

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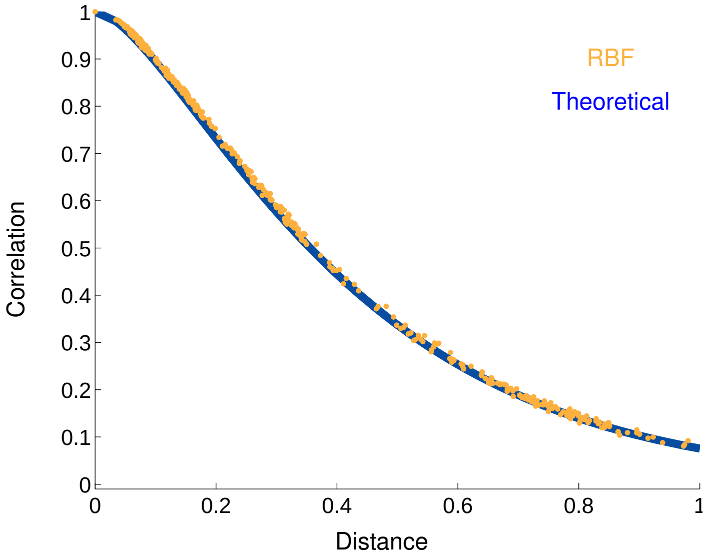

Conclusions - Future work

- Conclusions
  - RBFs can extend the SPDE approach to higher dimensions
  - Avoid the triangulation cost and overhead
  - Precision matrix is not (likely) to be sparse
  - Solution is sensitive to the selection of the RBF width
- Future work
  - Estimate parameters from data
  - Extend to non-stationary version of the SPDE

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Reference

- Lindgren, F., Rue, H. and Lindström, J., '*An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach*', J. R. Statist. Soc. B, (73):2011