Betti numbers and minimal free resolutions for multi-state system reliability bounds

ABSTRACT

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The paper continues work on monomial ideals in system reliability began by Giglio and Wynn [GW04] following work in discrete tube theory by Naiman and Wynn [NW92, NW97]. The key component is that of multigraded Betti numbers, and an algorithm using Mayer-Vietoris trees by the first author [dC06] is the main tool.

First a mapping must be made between the states of a multistate system and a monomial ideal, or more specifically a collection of monomials. A multi-state system is a system of n components whose states are described by real variables $Y = (Y_1, \ldots, Y_n)$. The (discrete) states of each system are labeled by $\{1, 2, \ldots\} = \mathbb{N}$ so that $\mathcal{Y} = \mathbb{N}^n$. Then $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{Y}$ is encoded by $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$.

Assume that the system has a distinguished subset $\mathcal{F}$, called the failure set which is such that for $\mathbf{a} \leq \mathbf{b}, \mathbf{a} \in \mathcal{F} \Rightarrow \mathbf{b} \in \mathcal{F}$. Then the system is said to be *coherent*: if the system fails, in the sense of its state being in $\mathcal{F}$, it will also fail at a more extreme state. The main conceptual link between the two fields is that this correspond exactly to the monomial ideal property. Thus, if $Id_{\mathcal{F}} = \langle x^{\mathbf{a}} : \mathbf{a} \in \mathcal{F} \rangle$ then $\mathbf{a} \leq \mathbf{b}, x^{\mathbf{a}} \in Id_{\mathcal{F}} \Rightarrow x^{\mathbf{b}} \in Id_{\mathcal{F}}$. In addition we may consider the minimal cut set $\mathcal{F}^*$ which corresponds to the minimal generators of the ideal $Id_{\mathcal{F}}$.

If the behaviour of the system is described by allowing the state to be the realisation of a random variable Y, then the failure probability is $\text{prob}\{Y \in \mathcal{F}\}$. If we can find bounds or equalities for the indicator function of $\mathcal{F}$ then they are inherited by this probability. Equivalently, here, we bound the generating function: $\mathcal{F}(x) = \sum_{\mathbf{a} \in \mathcal{F}} x^{\mathbf{a}}$.

Consider a multigraded R-module, $\mathcal{M}$, over the ring $R = k[x_1, \ldots, x_n]$ considered as a k vector space over each of its multigraded "pieces", and a monomial ideal I. If an R-resolution $\mathcal{P}$ of I is multigraded we obtain the *muligraded Hilbert series* is given by

$$\mathcal{H}(R/I; x) = \frac{\sum_{i=0}^{d} (-1)^i (\sum_{\alpha \in \mathbb{N}^n} \gamma_{\alpha,i} \cdot x^{\alpha})}{\prod_{j=1}^{n} (1 - x_i)},$$

where the $\gamma_{\alpha,i}$ are the ranks of the multigraded piece of degree α in the i-th module of $\mathcal{P}$, $\mathcal{P}_i$. If, furthermore, the resolution is minimal then

$$\mathcal{H}(R/I; x) = \frac{\sum_{i=0}^{d} (-1)^i (\sum_{\alpha \in \mathbb{N}^n} \beta_{\alpha,i} \cdot x^{\alpha})}{\prod_{j=1}^{n} (1 - x_i)},$$

where $\beta_{\alpha,i}$ are the multigraded Betti numbers of I and

$$\beta_{\alpha,i} \leq \gamma_{\alpha,i} \quad \forall \alpha, i \tag{1}$$

The central idea is that, from (1), if we set $I = Id_{\mathcal{F}}$ and truncate the Hilbert series (i) we obtain upper and lower bounds for the Hilbert series and (ii) for a minimal resolution these bounds are at least as tight as for any other resolution:

$$\frac{\sum_{i=1}^{k+1}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\gamma_{\alpha,i}\cdot x^{\alpha})}{\prod_i(1-x_i)} \leq \frac{\sum_{i=1}^{k+1}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\beta_{\alpha,i}\cdot x^{\alpha})}{\prod_i(1-x_i)} \leq \mathcal{H}(I;x)$$
$$\leq \frac{\sum_{i=1}^{k}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\beta_{\alpha,i}\cdot x^{\alpha})}{\prod_i(1-x_i)} \leq \frac{\sum_{i=1}^{k}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\gamma_{\alpha,i}\cdot x^{\alpha})}{\prod_i(1-x_i)} \quad (2)$$

$k = 1, \ldots, d-1; k$ odd.

The inner inequalities in (2) give, over all resolutions the tightest inclusion exclusion bounds for the system's reliability for $\mathcal{F}(x)$. The standard inclusion exclusion bounds are given by the Taylor complex, in which we use all all "index sets":

$$\mathcal{H}(Id_{\mathcal{F}},x) = \frac{\sum_{j=1}^{r}(-1)^{j-1}\sum_{|J|=j}m_J}{\prod_i(1-x_i)},$$

Other resolutions include the Scarf complex [MS04] which is minimal under a genericity condition, and was already used in [GW04].

The paper considers a number of examples in which we can directly compute the multigraded Betti numbers of $Id_{\mathcal{F}}$, without necesarily computing the minimal free resolution. The techniques for these computations include simplicial Koszul complexes [Bay96] and Mayer-Vietoris trees [dC06]. Both methods make use of the equality between the Betti numbers and the dimension of the Koszul homology modules, which comes from the equivalent ways of computing $Tor_{\bullet}(\mathbf{k}, I)$ for any ideal $I \subseteq \mathbf{k}[x_1, \ldots, x_n]$ either using resolutions of I or resolutions of $\mathbf{k}$, such as the Koszul complex $\mathbb{K}(I)$ (see [dC06]). The paper considers three examples, two special families of systems and a general class of networks:

k-out-of-n systems. These are generated by all square free monomials in n variables of a given degree k, such as $I_{3,5} = \langle xyz, xyu, xyv, xzu, xzv, xuv, yzu, yzv, yuv, zuv\rangle$. The Koszul complex can be completely described and the multigraded Betti numbers have a closed combinatorial form.

Consecutive k-out-of-n. Here the monomials generating the ideal are also square free but they have the variables adjacent in sequence e.g. $\overline{I}_{3,5} = \langle xyz, yzu, zuv\rangle$. This example is more complex but the lexicographic Mayer-Vitoris tree can be used to find the multigraded Betti numbers. We prove that there are no repeated multidegree exponents in the relevant node of this tree. This gives a fast recursive method for obtain the Betti numbers for given n and k.

Parallel-series systems. We define a mixed class called a *parallel-series network* as a network such that if either N consists of an input node, an output node and a edge joining them, or if $N = N_1 + N_2$ or $N = N_1 \times N_2$ with N_1, N_2 series-parallel networks. We prove a proposition that such a systems have corresponding ideals which are Mayer-Vietoris ideals of type A, i.e. the multigraded Betti numbers are abtained directly from their Mayer-Vietoris trees. In turns out that the "+" and "×" operations which are used to build up the network induce analogous operations in the construction of the tree and the corresponding monomial ideals.

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Introduction

The use of monomial ideals in system reliability was introduced by Giglio and Wynn [GW04] following work on so-called discrete tube theory by Naiman and Wynn [NW92] and [NW97]. Dohmen [Doh03] uses the latter work also to study reliability. The proof of the main result in [NW97] made use of arguments from algebra, particularly Betti numbers. Also in [GW04], where Scarf resolutions were used, it was suggested that minimal free resolutions should be sought. In this context, reliability bounds are given by multigraded Hilbert functions and series of the correspondent monomial ideals, which can be read from free resolutions. Sharper bounds are obtained via minimal resolutions, and are given by the alternating sums of the ranks of the multigraded pieces of their modules, i.e. the multigraded Betti numbers. For computing these, several tools are available, including simplicial Koszul complexes [Bay96, MS04] and Mayer-Vietoris trees [dC06].

1 System reliability

A multi-state system is defined here as a system of n components whose states are described by real variables $Y = (Y_1, \ldots, Y_n)$, which can be in one of a set of states which we define as the n-dimensional non-negative integer grid $\mathcal{Y} = \mathbb{N}^n$. There is a distinguished subset, $\mathcal{F} \subset \mathcal{Y}$, called the *failure set*, with the interpretation that if $Y \in \mathcal{F}$ the system is said to fail. A member of $\mathcal{F}$ is called a *cut*. Let $\leq$ be the usual multivariate inequality $y \leq z \Leftrightarrow y_i \leq z_i,\ i = 1, \ldots, n$ and let $y < z$ when $y \leq z$ and $y_i < z_i$ for at least one $i = 1, \ldots, n$. Also define $x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$. Then we call the system *coherent* if

$$y \in \mathcal{Y},\ y \leq z \Rightarrow z \in \mathcal{Y} \tag{1}$$

Note that we use y to refer to a particular value (point) in $\mathcal{Y}$ and use Y for the random variable describing the stochastic behaviour of the system. Coherency is the principle that if a system has failed and the components move to a worse (higher) state value then the system remains failed.

In reliability, Y is a random variable, which summarises the consequence of internal degradation or external shock to the system liable to increase the values of states, although by repair one can also decrease the value. Indeed, in Markovian systems one can consider Y moving around $\mathcal{Y}$ according to a Markov chain; see, for example, the study of maintenance systems.

A major concern of system reliability is to evaluate or bound the probability of failure $P(\mathcal{F}) = \text{prob}\{Y \in \mathcal{F}\}$. But we will be concerned, not so much with the dependence of $P(\mathcal{F})$ on the distribution of Y, but rather with the set $\mathcal{F}$ itself. Thus for any set $U \subseteq \mathcal{Y}$ we define the indicator

$$I_U(y) = \begin{cases} 1 & \text{if } y \in U \\ 0, & \text{otherwise} \end{cases}$$

Then $P(\mathcal{F}) = \mathrm{E}\,(I_U(Y))$ and identities and bounds on indicator functions give identities and bounds on $P(\mathcal{F})$, whatever the distribution of Y.

2 Monomial ideals

The first step in the algebraization of coherent systems is to encode a point $\alpha = (\alpha_1, \dots \alpha_n) \in \mathcal{Y}$ by a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $x = (x_1, \dots, x_n)$ is a vector of variables. We see immediately from the coherence property (1) that $\mathcal{Y} = \mathbb{N}^n$ is coded into a set of monomials which defines a monomial ideal

$$Id_{\mathcal{F}} = \langle x^\alpha : \alpha \in \mathcal{F} \rangle$$

and (1) is equivalent to the ideal property

$$x^\alpha \in Id_{\mathcal{F}},\ \alpha \leq \beta \ \Rightarrow\ x^\beta \in Id_{\mathcal{F}}.$$

Conversely, any monomial ideal gives a failure set, under coherency. The minimal basis for the monomial ideal $Id_{\mathcal{F}}$ can be identified with the set, $\mathcal{F}^*$, of *minimal cuts*, in the reliability context. Thus α is a minimal cut if and only if $\alpha \in \mathcal{F},\ \beta < \alpha \Rightarrow \beta \notin \mathcal{F}$ and moreover $Id_{\mathcal{F}} =< x^\alpha \mid \alpha \in \mathcal{F}^* >$.

A subset $U \subset \mathcal{Y}$ has a unique generating function:

$$U(x) = \sum_{\alpha \in U} x^\alpha,$$

and identities and inequalities on their indicator functions, $I_U(y)$ can be translated precisely to those for the corresponding generating functions. In particular we shall be interested in identities and bounds for $\mathcal{F}(x)$, the generating function of the failure set $\mathcal{F}$. The generating function for the whole of $\mathcal{Y} = \mathbb{N}^n$ and for the monomial ideal generated by a single monomial are respectively

$$\begin{aligned} \mathcal{Y}(x) &= \frac{1}{\prod_{i=1}^n (1-x_i)}, \\ \{\beta\}(x) &= \frac{x^\beta}{\prod_{i=1}^n (1-x_i)} \end{aligned}$$

As an example, consider just two minimal cuts, $\mathcal{F}^* = \{\beta, \gamma\}$. Then the failure ideal is $Id_{\mathcal{F}} =< x^\beta, x^\gamma >$, and the generating function of the associated monomial set is

$$\mathcal{F}(x) = \frac{x^\alpha + x^\beta - \text{lcm}(x^\alpha, x^\beta)}{\prod_{i=1}^n (1 - x_i)} = \{\alpha\}(x) + \{\beta\}(x) - \{\alpha \vee \beta\}(x) \qquad (2)$$

This represents inclusion-exclusion for the failure set of the relevant upper orthants in the original system $\mathcal{Y}$:

$$I_{Q(\alpha) \cup Q(\beta)} = I_{Q(\alpha)}(y) + I_{Q(\beta)}(y) - I_{Q(\alpha) \cap Q(\beta)}(y) = I_{Q(\alpha)}(y) + I_{Q(\beta)}(y) - I_{Q(\alpha)}(y) I_{Q(\beta)}(y),$$

where $Q(\alpha) = \{\beta | \alpha \leq \beta\}$, etc are the orthants. Note that if we omit the last term on the right hand side we obtain an upper bound to the indicator function which gives the elementary Bonferroni bound: $\text{prob}(Q(a) \cup Q(b)) \leq \text{prob}(Q(a)) + \text{prob}(Q(b))$.

3 Improved bound via the multigraded Hilbert Series

Consider a multigraded R-module, $\mathcal{M}$, over the ring $R = k[x_1, \ldots, x_n]$ considered as a k vector space over each of its multigraded "pieces". If each of the dimensions is finite we can define the *multigraded Hilbert series* as the formal power series

$$\mathcal{H}(\mathcal{M}; x) = \sum_{\alpha \in \mathbb{N}^n} dim_k(\mathcal{M})_\alpha x^\alpha$$

For a resolution of the quotient of R by monomial ideal I we have, from the rank-nullity principle, that

$$\mathcal{H}(R/I; x) = \sum_{i=0}^{d} (-1)^i \mathcal{H}(P_i; x),$$

where the P_i, $i = 0, \ldots, d$ are the modules in the resolution of R/I. If the resolution is multigraded each $P_i = \bigoplus_{\alpha \in \mathbb{N}^n} \gamma_{\alpha,i} P_{\alpha,i}$ for scalars $\gamma_{\alpha,i}$, of which only a finite number are non-zero. Then

$$\mathcal{H}(R/I; x) = \frac{\sum_{i=0}^{d} (-1)^i (\sum_{\alpha \in \mathbb{N}^n} \gamma_{\alpha,i} \cdot x^\alpha)}{\prod_{j=1}^{n} (1 - x_i)}$$

If the resolution is minimal then

$$\mathcal{H}(R/I; x) = \frac{\sum_{i=0}^{d} (-1)^i (\sum_{\alpha \in \mathbb{N}^n} \beta_{\alpha,i} \cdot x^\alpha)}{\prod_{j=1}^{n} (1 - x_i)},$$

where $\beta_{\alpha,i}$ are the multigraded Betti numbers and, importantly,

$$\beta_{\alpha,i} \leq \gamma_{\alpha,i} \quad \forall \alpha, i \tag{3}$$

When $I = Id_{\mathcal{F}}$ the Hilbert series of I and R/I are, respectively, the generating functions of $\mathcal{F}$ and $\mathcal{Y} \setminus \mathcal{F}$, the latter being the non-failure set (where the systems works), and

$$H(I; x) = \frac{\sum_{i=1}^{d} (-1)^{i-1} (\sum_{\alpha \in \mathbb{N}^n} \gamma_{\alpha,i} \cdot x^\alpha)}{\prod_i (1 - x_i)}$$

The key idea for system reliability is that if we truncate this multigraded Hilbert series, using exactness and the optimality (3), (i) we obtain upper and lower bounds for the Hilbert function and (ii) for a minimal resolution these bounds are at least as tight as for any other resolution:

$$\frac{\sum_{i=1}^{k+1} (-1)^{i-1} (\sum_{\alpha \in \mathbb{N}^n} \gamma_{\alpha,i} \cdot x^\alpha)}{\prod_i (1 - x_i)} \leq \frac{\sum_{i=1}^{k+1} (-1)^{i-1} (\sum_{\alpha \in \mathbb{N}^n} \beta_{\alpha,i} \cdot x^\alpha)}{\prod_i (1 - x_i)} \leq \mathcal{H}(I; x)$$
$$\leq \frac{\sum_{i=1}^{k} (-1)^{i-1} (\sum_{\alpha \in \mathbb{N}^n} \beta_{\alpha,i} \cdot x^\alpha)}{\prod_i (1 - x_i)} \leq \frac{\sum_{i=1}^{k} (-1)^{i-1} (\sum_{\alpha \in \mathbb{N}^n} \gamma_{\alpha,i} \cdot x^\alpha)}{\prod_i (1 - x_i)} \tag{4}$$

$k = 1, \ldots, d - 1; k$ odd.

3.1 Different resolutions

Let $\mathcal{F}$ be the failure set for a coherent system and label its elements $\mathcal{F}^* = \{\alpha^{(i)}, i = 1, \ldots, r\}$. For an index set $J \subset \{1, \ldots, r\}$ define $m_J = \text{lcm}\{x^{\alpha^{(j)}}, j \in J\}$. Then the classical inclusion exclusion lemma corresponds to the Taylor resolution and we can write the generating function, equivalently Hilbert series, as

$$\mathcal{H}(Id_{\mathcal{F}}, x) = \frac{\sum_{j=1}^{r}(-1)^{j-1}\sum_{|J|=j} m_J}{\prod_i (1 - x_i)},$$

Since the minimal resolution is a subresolution of the Taylor resolution, from (4) we can claim that truncated inclusion-exclusion bounds based on minimal free resolutions is at least as good as the truncated inclusion-exclusion bounds, sometimes referred to as generalised Bonferroni bounds.

It may be that we have repetitions of m_j in the Taylor complex. A simplicial complex similar to the Taylor complex in construction but which is restricted to unique labels ($m_I = m_J \Rightarrow I = J$) is the Scarf complex (see [MS04]). If in addition the generators $x^\alpha, x \in \mathcal{F}^*$ are in generic position (no variable x_i appears with the same (non-zero) exponent in two distinct generators) then the Scarf complex gives a minimal free resolution of $Id_{\mathcal{F}}$. There are a number of other types of resolutions, including Lyubeznik resolution, cellular resolutions, or the recent resolutions constructed via frames and degenerations [PV07], but the efficient computation of the minimal free resolution of a monomial ideal is in general a complicated task. Some methods for the computation of multigraded Betti numbers are described in the next section.

4 Computation of multigraded Betti numbers of monomial ideals

Since we are interested in the multigraded Betti numbers of $Id_{\mathcal{F}}$, we can use methods that compute them without necesarily computing the minimal free resolution. These include simplicial Koszul complexes [Bay96] and Mayer-Vietoris trees [dC06], which, in addition to its general algebraic uses, appear to be effective for certain types of problems in reliability. Both methods make use of the equality between the Betti numbers and the dimension of the Koszul homology modules, which comes from the equivalent ways of computing $Tor_\bullet(\mathbf{k}, I)$ for any ideal $I \subseteq \mathbf{k}[x_1, \ldots, x_n]$ either using resolutions of I or resolutions of $\mathbf{k}$, such as the Koszul complex $\mathbb{K}(I)$ (see [dC06]).

4.1 Simplicial Koszul complexes

Definition 4.1 *Let I be a monomial ideal, $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $x^\alpha \in I$. The* Koszul simplicial complex*, is given by*

$$\Delta^I_\alpha = \{\textit{squarefree vectors } \tau | \mathbf{x}^{\alpha-\tau} \in I\}$$

With this definition we have the following result that relates the simplicial homology of the Koszul simplicial complex to the multidegree α Betti numbers of I (see [Bay96],[MS04]).

Theorem 4.2
$$\beta_{i,\alpha}(I) = dim(H_{i,\alpha}(\mathbb{K}(I))) = dim(\widetilde{H}_{i-1}(\Delta^I_\alpha)) \quad \forall i$$

If we call L_I to the lcm-lattice of $I = \langle m_1, \ldots, m_r \rangle$, i.e. the lattice with elements labeled by the least common multiples of subsets of $\{m_1, \ldots, m_r\}$ ordered by divisibility, we have that
$$\beta_{i,\alpha}(I) = 0 \text{ if } \alpha \notin L_I \tag{5}$$
Therefore, to compute the dimensions of the multigraded Koszul homology modules of I, i.e. the multigraded Betti numbers of I, we need only compute the dimensions of the homology of the simplicial Koszul complexes at the points in L_I, which is a finite set.

4.2 Mayer-Vietoris trees

Given a monomial ideal I minimally generated by $\{m_1, \ldots, m_r\}$, we can construct an analogue of the well known Mayer-Vietoris sequence from topology, in the following way:

Definition 4.3 *For each $1 \leq s \leq r$ denote $I_s := \langle m_1, \ldots m_s \rangle$, $\tilde{I}_s := I_{s-1} \cap \langle m_s \rangle = \langle m_{1,s}, \ldots, m_{s-1,s} \rangle$, where $m_{i,j}$ denotes $lcm(m_i, m_j)$. Then, for each s we have*

$$\cdots \longrightarrow H_{i+1}(\mathbb{K}(I_s)) \xrightarrow{\Delta} H_i(\mathbb{K}(\tilde{I}_s) \longrightarrow$$
$$H_i(\mathbb{K}(I_{s-1}) \oplus \mathbb{K}(\langle m_s \rangle)) \longrightarrow H_i(\mathbb{K}(I_s)) \xrightarrow{\Delta} \cdots \tag{6}$$

And since the Koszul differential respects multidegrees, we also have a multigraded version of the sequence.

Using recursively these exact sequences for every $\alpha \in \mathbb{N}^n$ we could compute the Koszul homology of $I = \langle m_1, \ldots, m_r \rangle$. The involved ideals can be displayed as a tree, the root of which is I and every node J has as *children* $\tilde{J}$ on the left and J' on the right (if J is generated by r monomials, $\tilde{J}$ denotes $\tilde{J}_r$ and J' denotes J_{r-1}). This is what we call a **Mayer-Vietoris Tree** of the monomial ideal I, and we will denote it $MVT(I)$. Each node in a Mayer-Vietoris tree is given a *position*: the root has position 1 and the left and right children of the node in position p have respectively, positions $2p$ and $2p+1$. The node in position p is denoted $MVT_p(I)$.

Remark 4.4 *We can sort the generators in a node J of a Mayer-Vietoris trees in many different ways, and for each such sorting there is a different Mayer-Vietoris tree. For simplicity of notation, we assume we have the generators already sorted and use the last generators to obtain the ideals $\tilde{J}$ and J'. In fact we only need a strategy to select a monomial in the node, which acts as a "pivot" for the construction of the tree. If we sort all the nodes using a term order, for example, lexicographic, and use the last monomial in each node as "pivot", we say we build the lexicographic tree, and so on.*

The properties of Mayer-Vietoris trees allow us to perform Koszul homology computations using them (see the details in [dC06]).

Proposition 4.5 *If $H_{i,\alpha}(\mathbb{K}(I)) \neq 0$ for some i, then x^α is a generator of some node J in any Mayer-Vietoris tree $MVT(I)$.*

Thus, all the multidegrees of Koszul generators (equivalently Betti numbers) of I appear in $MVT(I)$. For a sufficient condition, we need the following notation: among the nodes in $MVT(I)$ we call *relevant nodes* those in an even position or in position 1.

Proposition 4.6 *If x^α appears only once as a generator of a relevant node J in $MVT(I)$ then there exists exactly one generator in $H_*(\mathbb{K}(I))$ which has multidegree α.*

The dimension of the homology to which relevant multidegrees contribute, can also be read from their position in the tree.

4.2.1 Mayer-Vietoris ideals

Let I be a monomial ideal and $MVT(I)$ a Mayer-Vietoris tree of I. Let $\alpha \in \mathbb{N}^n$; let $\overline{\beta_{i,\alpha}}(I) = 1$ if α is the multidegree of some non repeated generator in some relevant node of dimension i in $MVT(I)$ and $\overline{\beta_i}(I) = 0$ in other case. Let $\widehat{\beta_{i,\alpha}}(I)$ be the number of times α appears as the multidegree of some generator of dimension i in some relevant node in $MVT(I)$. Then for all $\alpha \in \mathbb{N}^n$ we have

$$\overline{\beta_{i,\alpha}}(I) \leq \beta_{i,\alpha}(I) \leq \widehat{\beta_{i,\alpha}}(I)$$

Definition 4.7 *Let I be a monomial ideal.*

- *If there exists a Mayer-Vietoris tree of I such that there is no repeated generator in the ideals of the relevant nodes, then we say that I is a Mayer-Vietoris ideal of type A. In this case, $\overline{\beta_{i,\alpha}}(I) = \beta_{i,\alpha}(I) = \widehat{\beta_{i,\alpha}}(I)$ $\forall i \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$.*
- *If $\overline{\beta_{i,\alpha}}(I) = \beta_{i,\alpha}(I)$ for all $\alpha \in \mathbb{N}^n$ then we say that I is a Mayer-Vietoris ideal of type B1.*
- *If $\widehat{\beta_{i,\alpha}}(I) = \beta_{i,\alpha}(I)$ for all $\alpha \in \mathbb{N}^n$ then we say that I is a Mayer-Vietoris ideal of type B2.*

Remark 4.8 *It is not hard to show [dC06] that Mayer-Vietoris trees provide resolutions of the corresponding ideals. Therefore, the alternating sums of the bounds of the Betti numbers that are given by these trees provide reliability bounds in the sense exposed above. If the corresponding ideal is Mayer-Vietoris of type A or $B2$ then the resolution given by the Mayer-Vietoris tree is minimal. If it is of type $B1$, the minimal resolution is not directly obtained by the tree (we need to perform further computations to minimize it) but the multigraded betti numbers are immediately read from the tree, so sharp reliabilty bounds are also provided, observe that generic ideals are Mayer-Vietoris of type $B1$. In the other cases, the resolutions obtained by the tree are not minimal in general, but their size is relatively small (see examples in [dC06]) and therefore the reliability bounds provided by Mayer-Vietoris trees are fairly good in average for general ideals.*

5 Special examples in reliability

Classical system reliability deals with two-state or binary systems in which $\mathcal{Y} = \{0,1\}^d$: every component can fail or not fail. Because in general such systems are not generic the minimal resolution cannot be derived from the Scarf complex and some kind of algorithm to find the minimal resolution must be used. In [2] a special perturbation method was used. A starting point for the present collaboration was made when it transpired that some of the examples in that paper were indeed minimal resolutions and some not. It pointed to systematic application of a minimal free resolution method to reliability. We begin with two classical problems, k-out-of-n and consecutive k-out-of-n systems and then address an important class of problem at the heart of reliability theory namely series and parallel systems. In these problems our aim is always to derive the multigraded Betti numbers which give the optimal bounds in the sense of (4). The results may be purely computational, for example in some complex case, or may lead to a theoretical result in which the Betti numbers can be given a closed form or be related to the structure of the problem in some way.

5.1 k-out-of-n systems

A k-out-of-n system is one in which if at least k out of a total of n components fail then the system is said to fail. There is a considerable literature in the area within reliability but it may first have arisen in the context of occupancy problems and is covered in the classical text by Feller [Fel71] the first edition which was 1950 and contains a footnote to M. Frechet. The formula in [Fel71] Chapter IV Section 5 is exactly as derived here by our methods.

A k-out-of-n system can be modeled by the ideal

$$I_{k,n} = \langle x^\mu : x^\mu \text{ is a squarefree monomial of degree } k \text{ in } n \text{ variables}\rangle$$

for example, $I_{3,5} = \langle xyz, xyu, xyv, xzu, xzv, xuv, yzu, yzv, yuv, zuv\rangle$ is the ideal corresponding to the 3-out-of-5 problem. Observe that $I_{k,n}$ has a minimal generating set which consists of $\binom{n}{k}$ monomials. Using the result pointed in equation (5), we know that we have to check the Koszul homolgy only in the multidegrees that are in the lcm-lattice of I, namely L_I. It is easy to see that L_I consists of all squarefree monomials involving a number of variables between k and n. The following lemma caracterizes the Koszul simplicial complex at each of these multidegrees:

Lemma 5.1 *If $\alpha \in L_I$ has $k+i$ nonzero indices, $k < k+i \leq n$, the simplicial Koszul complex $\Delta_\alpha^{I_{k,n}}$ consists of all j-faces with $0 \leq j \leq i-1$.*

Proof: Let $\mathbf{x}^\alpha$ be a squarefree monomial consisting of the product of $k+i$ variables, $k < k+i \leq n$. If we divide $\mathbf{x}^\alpha$ by the product of j of these variables then: If $j \leq i$ then the resulting monomial is the product of a set of $k+i-j$ variables, and thus, a $j-1$ face is present in the Koszul simplicial complex. If $j > i$ then the result of the division is the product of $k+i-j$ variables, being $j > i$, $k+i-j < k$ and thus this product is not in $I_{k,n}$, so no $j-1$ face is in the simplicial Koszul complex for $j > i$.□

Thus, the (α, i)-th Betti number at the multidegree given by any combination of $k+i$ variables is $dim(\tilde{H}_{i-1}(C_{k,i}))$, where $C_{k,i}$ is the subcomplex of the $k+i$ dimensional simplex Δ_{k+i} having as facets all the $(i-1)$-faces. And then, $\beta_i(I_{k,n}) = \binom{n}{k+i} \cdot dim(\tilde{H}_{i-1}(C_{k,i}))$, for all $i \in \{0, \ldots, n-k\}$.

Our next goal is then to compute the dimension of the reduced homology of the complexes $C_{k,i}$. Since all faces in dimension less or equal $i-1$ are present in the complex, we know that $C_{k,i}$ has zero homology at all dimensions except possibly at dimension $i-1$. The chain complex of $C_{k,i}$ has the following form:

$$0 \to C_{i-1} \stackrel{\delta_{i-1}}{\to} \cdots \to C_1 \stackrel{\delta_1}{\to} C_0 \to 0$$

we have $\tilde{H}_j(C_{k,i}) = 0 \, \forall j < i-1$ thus $ker\,\delta_j / im\,\delta_{j+1} = 0$ and $dim(ker\,\delta_j) = dim(im\,\delta_{j+1})$ for all $j < i-1$. On the other hand, we have the usual equality

$$dim(ker\,\delta_j) = dim(C_j) - dim(im\,\delta_j)$$

putting these together we have that

$$dim(\tilde{H}_{i-1}(C_{k,i}) = dim(ker\,\delta_{i-1}) = \binom{k+i}{i-1} - \binom{k+i}{i-2} + \cdots + (-1)^{i-2}\binom{k+i}{1} + (-1)^{i-1}$$

We can use now the following combinatorial identity:

$$\binom{k+i}{i-1} - \binom{k+i}{i-2} + \cdots + (-1)^{i-2}\binom{k+i}{1} + (-1)^{i-1} = \binom{i+k-1}{k-1}$$

and we obtain that for every $\alpha \in L_I$ where α is the product of $k+i$ variables, we have that

$$\beta_{(\alpha,i)}(I_{k,n}) = \binom{i+k-1}{k-1}$$

and since we have $\binom{n}{k+i}$ such an α, it follows that

$$\beta_i(I_{k,n}) = \binom{n}{k+i} \cdot \binom{i+k-1}{k-1} \qquad \forall 0 \leq i \leq n-k.$$

These considerations lead us to the following formula for the *multigraded Hilbert series* of I:

$$\mathcal{H}(I_{k,n}; x) = \frac{\sum_i (-1)^i \binom{i+k-1}{k-1} \cdot (\sum_{\alpha \in [n,k+i]} \cdot x^{\alpha})}{\prod_i (1-x_i)},$$

where $[n, k+i]$ denotes the set of $(k+i)$-subsets of $\{1, \ldots, n\}$.

Example 5.2 *For $I_{3,5}$ we have*

$$\mathcal{H}(R/I_{3,5}; \mathbf{x}) = \frac{1 - (xyz + xyu + xyv + xzu + xzv + xuv + yzu + yzv + yuv + zuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)}$$

$$+ \frac{3(xyzu + xyzv + xyuv + xzuv + yzuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)} - \frac{6(xyzuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)},$$

the Betti numbers of $I_{3,5}$ are then: $\beta_0 = 10$, $\beta_1 = 15$ and $\beta_2 = 6$.

Remark 5.3 *It is not hard to show that a k-out-of-n ideals is Mayer-Vietoris of type B2. Therefore, its Mayer-Vietoris tree provides the minimal resolution.*

5.2 Consecutive k-out-of-n systems

Consecutive, also called "sequential", k-out-of-n systems fail whenever at least k consecutive components in an ordered list of n components fail. It is also covered by Feller [Fel71] Chapter XIII. It is of some interest that Dohmen [Doh03] investigates them using a version of the methods in [3] and [4]. In addition to a significant literature within reliability the topic has received renewed interest because of its use in the fast detection of fluctuations in data streams using statistics collected from windows of data: so-called "scan statistics"; see Glaz, Naus and Wallenstein [GNW01]. In the probability literature the emphasis is in computing probabilities under given distributional assumptions, whereas, as pointed out in the Section 1, the bounds we derive are distribution free.

Consecutive k-out-of-n systems can be modelled by the ideals

$$\bar{I}_{k,n} = \langle x^{\mu} : \mu \text{ is a squarefree monomial in } n \text{ variables formed by } k \text{ consecutive variables}\rangle.$$

For example, $\bar{I}_{3,5} = \langle xyz, yzu, zuv\rangle$ is the ideal corresponding to the consecutive 3-out-of-5 system. In order to find the multigraded Betti numbers and Hilbert series of $\bar{I}_{k,n}$ we will use its lexicographic Mayer-Vietoris tree. The explicit construction of this tree will give us the results we need. For greater clarity, we will denote the monomials by their exponents in brackets, e.g the monomial $x_1x_3x_6$ will be denoted by $[1, 3, 6]$, since we are dealing with squarefree monomials, this notation suffices.

MVT($\bar{\mathbf{I}}_{\mathbf{k,n}}$)

We sort the generators of $\bar{I}_{k,n}$ using the lexicografic order. The construction of $MVT(\bar{I}_{k,n})$ is as follows:

1. The root node is just $\bar{I}_{k,n}$, which is minimally generated by $n - k + 1$ monomials.

2. The right child of the root, i.e. $MVT_3(\bar{I}_{k,n})$ is $\bar{I}_{k,n-1}$, so we hang here the corresponding tree.

3. The left child of the root, $MVT_2(\bar{I}_{k,n})$, consists of the following $n-2k+1$ monomials:

   $[j, \cdots (j+k-1), (n-k+1), \cdots, n]$ for $1 \leq j \leq n-2k$ which are the least common multiples of each of the first $n - 2k$ generators of the root with the last one. These generators have $2k$ variables.

   $[n - k, \cdots, n]$ which is the lcm of the last two generators of $MVT(\bar{I}_{k,n})_1$ and divides $[n - k - j, \cdots, n]$ for $1 \leq j \leq (k - 1)$ and hence these last will not appear as minimal generators of this node. This generator has $k + 1$ variables and since we are using lexicographic order, it will appear as the last generator in $MVT_2(\bar{I}_{k,n})$.

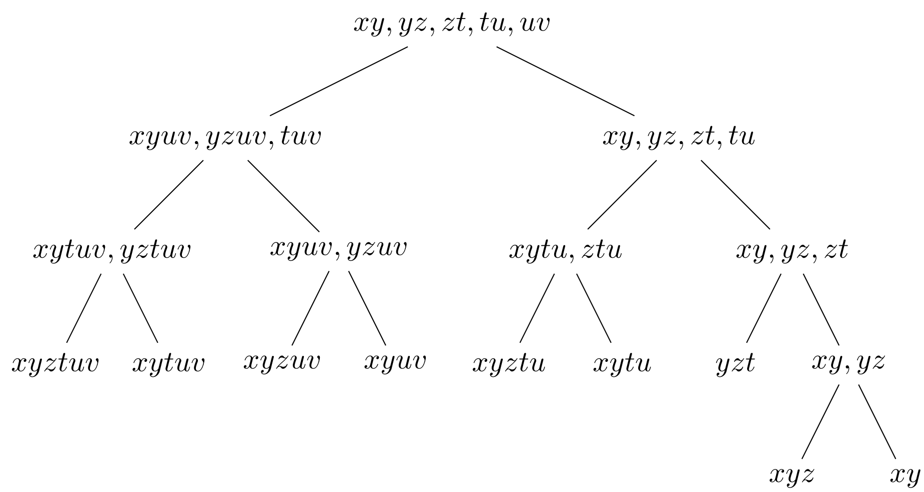
4. The following nodes to consider are $MVT_4(\bar{I}_{k,n})$ and $MVT_5(\bar{I}_{k,n})$, but only if $MVT_2(\bar{I}_{k,n})$ has more than one generator i.e. if $2k < n$, otherwise they are empty. If it is the case, then

   $MVT_4(\bar{I}_{k,n})$ consists of $n - 2k$ generators, namely the lcms of the first $n - 2k$ generators of $MVT_2(\bar{I}_{k,n})$ with the last one. These have the form $[j, \cdots (j + k - 1), (n - k), \cdots, n]$ for $1 \leq j \leq n - 2k$ and hence, this node is exactly equal to $\bar{I}_{k,n-k-1}$

with each monomial in it multiplied by $[n-k,\cdots,n]$. Hence, we hang here a tree 'isomorphic' to $MVT(\bar{I}_{k,n-k-1})$.

$MVT_5(\bar{I}_{k,n})$ is completely analogous to $MVT_4(\bar{I}_{k,n})$ and hence equal to $\bar{I}_{k,n-k-1}$ but this time each monomial in it is multiplied by $[n-k+1,\cdots,n]$. Hence, we also hang here a tree isomorphic to $MVT(\bar{I}_{k,n-k-1})$. The trees we have hanging from the corresponding nodes are of the same form, except that they have less variables, in particular they are of the form $MVT(\bar{I}_{k,j})$ with $j<n$. Eventually, we will have the situation in which $2k \geq n$ and in this case, the left child of the root has only one generator, namely $[j-k,\ldots,j]$, and the right node is the consecutive k-out-of-$(j-1)$ tree, so we proceed in this manner until $j=k+1$.

Example 5.4 *Here is the tree corresponding to the consecutive 2-out-of-6 system:*



Taking into account the properties of the Mayer-Vietoris trees of these ideals, we see that we can read the multigraded Betti numbers directly from the tree:

Proposition 5.5 *The ideal corresponding to the consecutive k-out-of-n system is Mayer-Vietoris of type A.*

Proof: Assume we have $\bar{I}_{k,n}$ as the root of our tree, sorted with respect to lexicographic order, then the variable n appears only in the left child of the root, and it will appear in every multidegree of every node in the tree hanging from this node (see the construction above). Thus, no multidegree of the tree hanging from the left child will appear in the tree hanging from the right child, and vice-versa. If $2k \geq n$ then we are done, since the left node has just one generator, and the tree hanging from the right node is the one correspondng to the k-out-of-$(n-1)$ system. If the left child of the root has more than one generator, then we look at its children, $MVT_4(\bar{I}_{k,n})$ and $MVT_5(\bar{I}_{k,n})$. The generators of the first one are not present in any node seen so far, and all of them contain the variables $(n-k),\ldots,n$; moreover, every generator of the nodes of the tree hanging from it will have these variables. On the other hand, the variable $n-k$ does not appear in the generators of $MVT(\bar{I}_{k,n})_5$ hence, no multidegree of a generator in the tree hanging from it will appear in the tree hanging from $MVT(\bar{I}_{k,n})_4$ and viceversa. Finally, we see that no multidegree appearing in any relevant node of the tree hanging from $MVT(\bar{I}_{k,n})_5$ is in $MVT(\bar{I}_{k,n})_2$. We know that $MVT(\bar{I}_{k,n})_5$ is generated by the generators of $MVT(\bar{I}_{k,n})_2$ except the last one. Now, every generator of every node in the tree hanging from $MVT(\bar{I}_{k,n})_5$ will have at least $2k+1$ different variables, k of which will be $(n-k+1),\dots,n$ (see the construction of the tree), and on the other hand, the generators in $MVT(\bar{I}_{k,n})_2$ have at most $2k$ different variables. □

With this proposition we have that collecting all the generators of the relevant nodes in $MVT(\bar{I}_{k,n})$ we have the multigraded Betti numbers of $\bar{I}_{k,n}$ in this case, since no generator in the relevant nodes is repeated, we have that the Betti number at each multidegree is 1, every multidegree appears only once in the minimal resolution of the ideal. The description of the tree and its recursive construction give us also means to count how many multidegrees appear in each dimension (i.e. the Betti numbers) and which multidegrees are present. A thorough description of this process would be tedious, but it is not difficoult to obtain a complete list of the multidegrees of the Betti numbers, and hence, of the Hilbert series. however, here we only give an idea of the procedure; an algorithm has been implemented by the authors to generate this list. The main lines of the construction of this list of multidegrees are the following.

1. In dimension 0 collect all the generators of $\bar{I}_{k,n}$.

2. In dimension 1 collect all the multidegrees of the form $[j,\dots,j+k]$ for $1\leq j\leq(n-k)$. [1] Moreover, for $k\leq j\leq(n-k)$, add the multidegrees $[1,\cdots,k,(j+1),\cdots,(j+k)],\dots,[(j-k),\dots,(j-1),(j+1),\cdots,(j+k)]$.

3. For every dimension l add the corresponding multidegrees that appear in $\bar{I}_{k,j-k-1}$ in dimension $(l-2)\geq 0$ multiplied by $[(j-k),\dots,j]$ and the multidegrees that appear in $\bar{I}_{k,j-k-1}$ in dimension $(l-1)\geq 0$ multiplied by $[(j-k+1),\dots,j]$ for all $(2k+1)\leq j\leq n$

Example 5.6 *As we can see from the tree of $\bar{I}_{2,6}$, the Betti numbers are $\beta_0=5$, $\beta_1=7$, $\beta_2=4$, $\beta_3=1$. And the multigraded Hilbert series:*

$$\mathcal{H}(R/\bar{I}_{2,6};\mathbf{x})=\frac{1-(xy+yz+zt+tu+uv)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)}$$

$$+\frac{(xyuv+yzuv+tuv+xytu+ztu+yzt+xyz)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)}$$

$$-\frac{(xytuv+yztuv+xyzuv+xyztu)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)}$$

$$+\frac{(xyztuv)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)},$$

[1] Note that in the case $2k\geq n$ these are the only ones we have to add, and the corresponding formula is equivalent to the one appearing in [Doh03]

5.3 Series and parallel systems

We turn now to series-parallel system, a special although very natural type of networks. Consider a edge p joining two nodes I and O. We call such a network a *basic* series-parallel network. Consider now two series-parallel networks N_1 and N_2. We can connect them in series or in parallel, and the result is a series-parallel network. This is done in the following way:

- First, we rename the edges in each node so that each edge has a different label. If the edge p_S for some (possibly empty) set S of subindices is in network i we can rename it $p_{\{i\}\cup S}$. After this, we can still rename them just by counting them in lexicographic order.
- If the initial (input) node of N_i is labelled I_i and its final (output) node is labelled O_i for $i = 1, 2$, then the parallel union of N_1 and N_2, which we will denote $N = N_1 + N_2$ identifies I_1 and I_2 in one node I, which will be the initial node of N , and identifies O_1 and O_2 in one node O, which will be its final node.
- With the same notation as above, the series union of N_1 and N_2, which we will denote $N = N_1 \times N_2$ has as initial node I_1, as final node O_2, and identifies O_1 and I_2 in one intermediate node S.

We just formalize these considerations in the following definition of *series-parallel networks:*

Definition 5.7 *We say that a network N is a* parallel-series network *if either N consists of an input node, an output node and a edge joining them, or if $N = N_1 + N_2$ or $N = N_1 \times N_2$ with N_1, N_2 series-parallel networks.*

These constructions can be seen in figure 1, in which the label of the edge p_S is just S.

Now consider the ideals associated to these networks. It is clear that the ideal I_N of a network N with just one edge p_1 connecting two nodes I and O is just $I_N = \langle x_1\rangle$. The construction operations $+$ and $\times$ we have just seen, have their counterpart in the ideals of the resulting networks:

Proposition 5.8 *Let N_1 and N_2 be two networks the edges of which are labelled (after renaming as seen above) $p_1, \ldots, p_{n_1}$ and $p_{n_1+1}, \ldots, p_{n_1+n_2}$. Then,*

$$I_{N_1+N_2} = I_{N_1} + I_{N_2} \qquad I_{N_1\times N_2} = I_{N_1} \cap I_{N_2}$$

where $I_{N_1+N_2}$ and $I_{N_1\times N_2}$ are ideals in $\mathbf{k}[x_1, \ldots, x_{n_1+n_2}]$

Proof: We have that

$$I_N = \langle x_S | S = \{s_1, \ldots, s_{k_s}\} \text{ is a minimal connection in N}\rangle$$

Therefore, the minimal pathes in $N_1 + N_2$ are those of N_1 *and* those in N_2, and there is no mixture between them. Then, it is easy to see that the generating set of $I_{N_1+N_2}$ is just the union of the generating sets of I_{N_1} and I_{N_2}, each being generated in a different set of variables.

Now, the minimal paths of $N_1 \times N_2$ can be split into two parts, the first one being a minimal path from $I_{N_1\times N_2}$ to S and the second one being a minimal path between S and

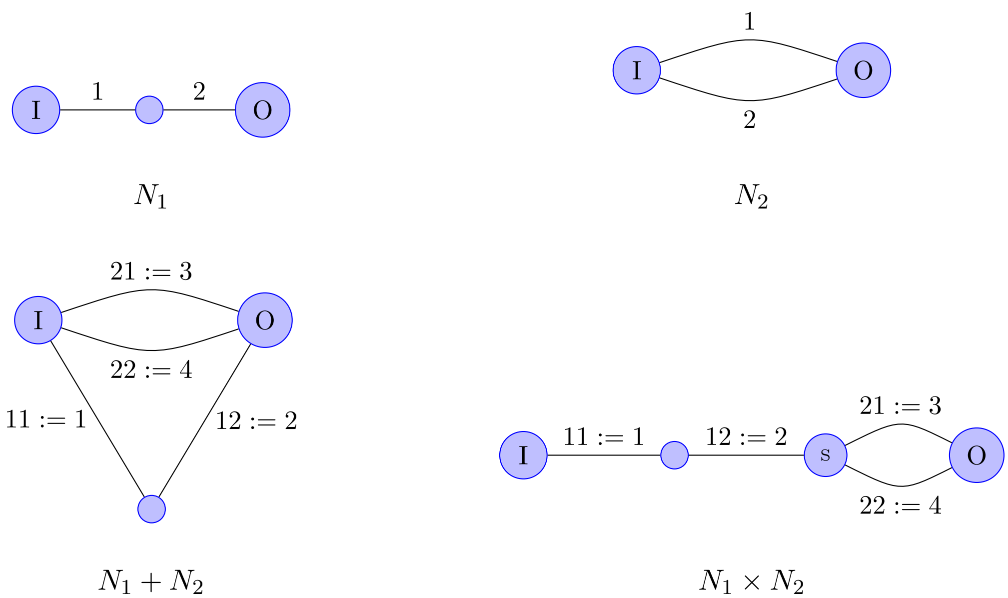


Figure 1: Example of series-parallel network construction.

$O_{N_1 \times N_2}$. Thus, each combination of one minimal path in N_1 and one minimal path in N_2 is a minimal path in $N_1 \times N_2$ and there are no other minimal pathes. Since there is no intersection between the set of variables of I_{N_1} and I_{N_2} the concatenation simply means a product, and hence the result. □

Example 5.9 *Consider the networks in figure 1, where* := *expresses relabelling. After relabelling, the edges in* N_1 *are* p_1 *and* p_2*, and the edges in* N_2 *are* p_3 *and* p_4*. We have that*

$$I_{N_1} = \langle x_1x_2 \rangle, \quad I_{N_2} = \langle x_3, x_4 \rangle, \quad I_{N_1+N_2} = \langle x_1x_2, x_3, x_4 \rangle, \quad I_{N_1 \times N_2} = \langle x_1x_2x_3, x_1x_2x_4 \rangle$$

Mayer-Vietoris trees give a good way to compute the multigraded Betti numbers of series-parallel ideals, and hence, the reliability of the corresponding network:

Proposition 5.10 *The ideals associated to series-parallel networks, i. e.* series-parallel ideals*, are Mayer-Vietoris ideals of type A.*

Proof: If N is a basic series-parallel network with unique edge p_1 then $I_N = \langle x_1 \rangle$ which is Mayer-Vietoris of type A. Now consider two series-parallel networks N_1 and N_2 whose ideals are Mayer-Vietoris of type A, i.e. there is some strategy for selecting the pivot monomials when constructing a Mayer-Vietoris tree such that it is of type A. We have to proof that $I_{N_1} + I_{N_2}$ and $I_{N_1} \cap I_{N_2}$ are Mayer-Vietoris of type A:

- The generators of $I_{N_1} + I_{N_2}$ are the union of the generating sets of I_{N_1} and I_{N_2}. We sort them so that the generators of I_{N_2} all appear after the generators of I_{N_1}. We now proceed taking as pivot monomial always a generator of I_{N_2} following the strategy used to build the minimal Mayer-Vietoris tree of I_{N_2}. Doing so, we have that $MVT_p(I_{N_1} + I_{N_2})$ has as generators the generators of I_{N_1} each one multiplied by some product of the variables of I_{N_2} and also the generators of $MVT_p(I_{N_2})$. So far, we have no repeated generators in the relevant nodes: Assume that there is some generator repeated in two relevant nodes at positions p and q. Then they have the same exponents in the variables of I_{N_1} and the same exponent in the variables of I_{N_2}. If the generator has only variables of the second ideal, to be equal would mean that they are equal in $MVT(I_{N_2})$. And since those generators with 'mixed variables' are all of the form $m \cdot m'$ with m a minimal generator of I_{N_1}, no two of these are repeated.

This procedure takes us to nodes in which no further element only in the variables of I_{N_2} is available. From this now on we follow on each node the strategy of $MVT(I_{N_1})$. Since these nodes in positions p have as generators all the minimal generators of I_{N-1} times some polynomial m'_p in the variables of the second ideal. And since the m'_p are different for different p, we have that al the trees hanging from these nodes are isomorphic to $MVT(I_{N_1})$, therefore, there's no repeated generator in the relevant nodes in each of them. There is also no repetition among the different 'copies'of $MVT(I_{N_1})$ because of each $m'{}_p$ is unique.

- $I_{N_1} \times I_{N_2}$. Let us denote by $m_1, \ldots, m_r$ the generators of I_{N_1}, and by $m'_1, \ldots, m'_s$ the generators of I_{N_2}. Then $I_{N_1 \times N_2} = I_{N_1} \cap I_{N_2}$ is generated by $\{m_i m'_j | i = 1, \ldots, r; j = 1, \ldots, s\}$. Every generator of a relevant node in $MVT(I_{N_1 \times N_2})$ is of the form $m_J m'_{J'}$ with J, J' subsets of $\{1, \ldots, r\}$ and $\{1, \ldots, s\}$ respectively. We sort these generators so that we can follow a strategy 'compatible' with the strategies of $MVT(I_{N_1})$ and $MVT(I_{N_1})$: assuming that the generators in each of these trees wer sorted in such a way that the last one is always the pivot monomial, we sort the generators in our new tree in the following way: $m_J m'_{J'}$ precedes $m_K m'_{K'}$ if m_J precedes m_K in $MVT(I_{N_1})$, or if $m_J = m_K$ and $m'_{J'}$ precedes $m'_{K'}$ in $MVT(I_{N_2})$. Since the variables in I_{N_1} are all different from the variables in I_{N_2}, and because of the 'compatible'strategy when constructing $MVT(I_{N_1 \times N_2})$, we have that if $m_J \cdot m'_{J'}$ is in a relevant node of $MVT(I_{N_1 \times N_2})$, the m_J is in a relevant node of $MVT(I_{N_1})$ and $m'_{J'}$ is in a relevant node of $MVT(I_{N_2})$. It is clear that any two generators $m_J \cdot m'_{J'}$ and $m_K \cdot m'_{K'}$ satisfy that $J \neq K$ and/or $J' \neq K'$. Therefore, if $m_J \cdot m'_{J'} = m_K \cdot m'_{K'}$ it is because $m_J = m_K$ *and* $m'_{J'} = m'_{K'}$, which is a contradiction. So, in any case, the new ideal is Mayer-Vietoris of type A. □

6 Conclusions

It has been a long standing challenge to obtain improved bounds of Bonferroni type in system reliability, with many different types of improvement being suggested. We have shown that, among a class of bounds of resolution type, which include the classical case (equivalent to the Taylor resolution), the minimal free resolution is optimal and moreover this resolution is completely described by the multigraded Betti numbers. The computation of these numbers is usually done via minimal free resolutions, but these are in general hard to compute. In certain important classes of systems, alternative methods, such us the one proposed by the first author, can be used to obtain the multigraded Betti numbers in a more efficient way. On one side that these alternative methods should be used for such situations, and on the other side, that algebraic techniques can and should be used in many cases to improve the bounds given in the literature on coherent systems.

We have studied three types of system: two rather special and one, the series-parallel systems which is rather more general. But there are many other systems or which is leading example is give by a general network. One immediate example is a general network: what are the multigraded Betti numbers for a general network? Is there a very fast algorithm which relates them to the incidence structure?

An advantage of the current methods is that they apply naturally to the multi-state coherent systems case which are less thoroughly covered in the reliability literature. Indeed, the key connection is to code a state by the exponent of a monomial ideal. A big challenge both from the point of algebra and reliability is to generalise the notion of coherency. This would require different "geometries" to be included from that of unions of upper orthants. Other geometries were used in the original work on discrete tubes, [NW92], [NW97] and include unions of balls or half-spaces.

The connection of the present work with that of Dohmen [Doh03] needs to be studied. In addition to his application of discrete tube theory to reliability that author makes interesting links with other areas of combinatorics such as lace expansions, chromatic numbers and the Whitney broken circuit theorem. It is likely that minimal free resolutions and multigraded Betti numbers will be found to play a role in those theories also.

As pointed out the bounds given here are distribution-free: they are independent of the distribution of the random variable Y defining the (stochastic) system. But where the distribution takes a particular form eg independent failure of components or, say, a Markov chain, it is to be hoped that there is synergy between the minimal bounds give here and the distributions. This may leading to useful formulae for failure probabilities in particular cases. In statistics and probability there is interest in extreme events, for example for testing some kind simple null hypotheses, such as independence. Our bounds may contribute to an asymptotic theory as the failure set is pushed outwards, so that the first few terms of the bounds give simple formulae. To put it more succinctly: do multigraded Betti numbers play a part in certain "large deviation" theories?

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